# Basic topology 1

Find the interior, closure, and boundary of the following set: $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

## Solution

The interior of $A$ is the set of all points $(x_0, y_0) \in A$ such that for some radius $r \in \mathbb{R}$, the open ball with center $(x_0, y_0)$ and radius $r$ is included in $A$. That is, $B((x_0, y_0), r) \subseteq A$.

In this exercise, the set $A$ is the open ball centered at $(0, 0)$ with radius 1. To define its interior, we take any point $(x_0, y_0) \in A$ and we have to think if we can assign a value to $r$ so that we can form a small ball around that point and such that this ball is completely contained in the set $A$.

For each point $(x_0, y_0)$ of $A$ we can create a small open ball such that it is completely contained within $A$. Stated more rigorously, the radius $r$ of the ball must be smaller than the difference between the radius 1 and the distance from the point $(0, 0)$ to the point $(x_0, y_0)$.

That is to say, $r < 1 - \|(x_0, y_0)\|$.[1]

**Therefore, we can conclude that all points of $A$ are in its interior, that is to say, $A^\circ = A$.**

The boundary of $A$ is the set of points $(x_0, y_0) \in \mathbb{R}^2$ such that for every real number $r$, the open ball with center $(x_0, y_0)$ and radius $r$ intersects with $A$ and also with the complement of $A$ (that is, all the elements that are not in $A$).

**The points that lie on the circumference of the center $(0, 0)$ and radius 1, that is to say, the border of $B((0, 0), r)$, is the boundary of $A$.**

The closure of $A$ is the set of all points $(x_0, y_0) \in \mathbb{R}^2$ such that for every real number $r \in \mathbb{R}$, the open ball with center $(x_0, y_0)$ and radius $r$ shares at least one point with $A$, that is to say $B((x_0, y_0), r) \cap A \neq \emptyset$.

To define the closure of $A$ let us take any point $(x_0, y_0) \in \mathbb{R}^2$ and consider if for any value of $r$ the ball $B((x_0, y_0), r)$ intersects with $A$. If we choose any point from $\mathbb{R}^2$ (we have three options, that it is inside of $A$, on the border of $A$, or outside), only the points that are inside or on the border of $A$ are such that for any radius $r$, $B((x_0, y_0), r) \cap A \neq \varnothing$.

**The points that are in $A$ and on its boundary are in the closure, since for every $(x_0, y_0) \in B((0, 0), 1)$ and for every $r \in \mathbb{R}$, $B((x_0, y_0), r) \cap A \neq \varnothing$. Therefore, we can say that $\overline{A} = B((x_0, y_0), r)$.**

[1] Remember that the norm of a vector, is its distance from the origin